Reasoning on bipolar preference problems

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Abstract. Real-life problems present several kinds of preferences. In this paper we focus on problems with both positive and negative preferences, that we call *bipolar problems*. Although seemingly specular notions, these two kinds of preferences should be dealt with differently to obtain the desired natural behaviour. We technically address this by generalizing the soft constraint formalism, and by considering the issue of the compensation between positive and negative preferences. We then suggest how constraint propagation and branch and bound can be adapted to deal with bipolar problems. An extended version of this paper is also submitted to the technical programme.

1 Introduction

Real-life problems present several kinds of preferences. In this paper we focus on problems with both positive and negative preferences, that we call *bipolar problems*. Parts of this paper have appeared in [3].

Positive and negative preferences could be thought as two symmetric concepts, but this does not happen in real scenarios. In fact, assume, for example, to have a scenario with two objects A and B. If we like both A and B, i.e., if we give to A and B positive preferences, then the overall scenario should be more preferred than having just A or B alone, and so the combination of such a preferences should give an higher positive preference. Instead, if we dislike both A and B, i.e., if we give to A and B negative preferences, then the overall scenario should be less preferred than having just A or B alone and so the combination of such a negative preferences should give a lower negative preference. When dealing with both kinds of preferences, it is natural to express also indifference, which means that we express neither a positive nor a negative preference over an object. A desired behaviour of indifference is that, when combined with any preference, it should not influence the overall preference.

Finally, besides combining positive preferences among themselves, and also negative preferences among themselves, we also want to be able to combine positive with negative preferences, allowing compensation. For example, if we have a meal with meat (which we like very much) and wine (which we don't like), then what should be the preference of the meal? To know that, we should be able to compensate the positive preference given to meat with the negative preference given to wine.

In this paper we start from the soft constraint formalism [2] based on c-semirings, to model negative preferences. We then extend it via a new structure, that models positive preferences and then we define a combination operator between positive and negative preferences to model preference compensation. Finally, we propose how to adapt constraint propagation and branch and bound techniques for finding optimal solutions of bipolar problems.

2 Background: semiring-based soft constraints

A soft constraint [2] is a classical constraint [5] where each instantiation of its variables has an associated value from a (totally or partially ordered) set. This set has two operations, which makes it similar to a semiring, and is called a c-semiring. A c-semiring is a tuple $(A, +, \times, \mathbf{0}, \mathbf{1})$ where: A is a set and $\mathbf{0}, \mathbf{1} \in A$; $+$ is commutative, associative, idempotent, **0** is its unit element, and **1** is its absorbing element; $\times$ is associative, commutative, distributes over $+$, $\mathbf{1}$ is its unit element and **0** is its absorbing element. Consider the relation $\leq_S$ over A such that $a \leq_S b$ iff $a + b = b$. Then: $\leq_S$ is a partial order; $+$ and $\times$ are monotone on $\leq_S$; **0** is its minimum and **1** its maximum; $(A, \leq_S)$ is a lattice and, $\forall\, a, b \in A$, $a + b = lub(a, b)$. Moreover, if $\times$ is idempotent, then $(A, \leq_S)$ is a distributive lattice and $\times$ is its glb. Informally, the relation $\leq_S$ gives us a way to compare the tuples of values and constraints. In fact, when we have $a \leq_S b$, we will say that *b is better than a*.

Given a c-semiring $S = (A, +, \times, \mathbf{0}, \mathbf{1})$, a finite set D (the domain of the variables), and an ordered set of variables V, a constraint is a pair $\langle def, con \rangle$ where $con \subseteq V$ and $def : D^{|con|} \rightarrow A$. Therefore, a constraint specifies a set of variables (the ones in con), and assigns to each tuple of values of D of these variables an element of A. A soft constraint satisfaction problem (SCSP) is just a set of soft constraints over a set of variables. For example, fuzzy CSPs [8, 6] are SCSPs that can be modeled by choosing the c-semiring $S_{FCSP} = ([0, 1], max, min, 0, 1)$ and weighted CSPs [2] are SCSPs that can be modeled by using $S_{WCSP} = (\Re^+, min, +, +\infty, 0)$.

3 Negative preferences

The structure we use to model negative preferences is exactly a c-semiring [2] as described in the previous section. In fact, in a c-semiring there is an element which acts as indifference, that is $\mathbf{1}$, since $\forall a \in A$, $a \times \mathbf{1} = a$ and the combination between negative preferences goes down in the ordering (in fact, $a \times b \leq a, b$), that is a desired property. This interpretation is very natural when considering, for example, the weighted semiring $(R^+, min, +, +\infty, 0)$. In fact, in this case the real numbers are costs and thus negative preferences. The sum of different costs is worse in general w.r.t. the ordering induced by the additive operator (that is, min) of the semiring. From now on, we will use a standard c-semiring to model negative preferences, denoted as: $(N, +_n, \times_n, \bot_n, \top_n)$.

4 Positive preferences

When dealing with positive preferences, we want two main properties to hold: combination should bring to better preferences, and indifference should be lower than all the other positive preferences. These properties can be found in the following structure.

Definition 1. *A positive preference structure is a tuple* $(P, +_p, \times_p, \bot_p, \top_p)$ *such that* P *is a set and* $\top_p, \bot_p \in P$*;* $+_p$*, the additive operator, is commutative, associative, idempotent, with* $\bot_p$ *as its unit element* $(\forall a \in P, a +_p \bot_p = a)$ *and* $\top_p$ *as its absorbing element* $(\forall a \in P, a +_p \top_p = \top_p)$*;* $\times_p$*, the multiplicative operator, is associative, commutative and distributes over* $+_p$ $(a \times_p (b +_p c) = (a \times_p b) +_p (a \times_p c))$*, with* $\bot_p$ *as its unit element and* $\top_p$ *as its absorbing element*[1].

[1] In fact, the absorbing nature of $\top_p$ can be derived from the other properties.

The additive operator of this structure has the same properties as the corresponding one in c-semirings, and thus it induces a partial order over P in the usual way: $a \leq_p b$ iff $a +_p b = b$. This allows to prove that $+_p$ is monotone over $\leq_p$ and that it is the least upper bound in the lattice $(P, \leq_p)$.

Theorem 1. *Given the positive preference structure $(P, +_p, \times_p, \bot_p, \top_p)$, consider the relation $\leq_p$ over P. Then, $\times_p$ is monotone over $\leq_p$ (i.e., $\forall a, b \in P$ s. t. $a \leq_p b$, then $a \times_p d \leq_p b \times_p d, \forall d \in P$) and $\forall a, b \in P$, $a \times_p b \geq_p a +_p b \geq_p a, b$.*

On the other hand, $\times_p$ has different properties w.r.t. $\times_n$. More precisely, the best element in the ordering $(\top_p)$ is now its absorbing element, i.e., when we combine any positive preference a with $\top_p$, we get $\top_p$ and thus a disappears. While the worst element $(\bot_p)$ is its unit element, i.e., when we combine any positive preference a with $\bot_p$, we get a. $\bot_p$ is the element modelling indifference when we combine any positive preference. These are exactly the desired properties for the combination of positive preferences and for indifference w.r.t. positive preferences. An example of a positive preference structure is $P_1 = (R^+, max, sum, 0, +\infty)$, where preferences are positive reals aggregated with sum and compared with max.

5 Bipolar preference structures

For handing both positive and negative preferences we propose to combine the two structures described in sections 4 and 3 in what we call a *bipolar preference structure*.

Definition 2. *A bipolar preference structure is a tuple $(N, P, +, \times, \bot, \Box, \top)$ where*
- $(P, +_{|P}, \times_{|P}, \Box, \top)$ *is a positive preference structure;*
- $(N, +_{|N}, \times_{|N}, \bot, \Box)$ *is a c-semiring;*
- $+ : (N \cup P)^2 \longrightarrow (N \cup P)$ *is s. t. $a_n + a_p = a_p, \forall a_n \in N$ and $a_p \in P$; this operator induces a partial ordering on $N \cup P$: $\forall a, b \in P \cup N$, $a \leq b$ iff $a + b = b$;*
- $\times : (N \cup P)^2 \longrightarrow (N \cup P)$ *is an operator that, $\forall\ a, b, c \in N \cup P$, satisfies commutativity ($a \times b = b \times a$) and monotonicity property (if $a \leq b$, $a \times c \leq b \times c$).*

Bipolar preference structures generalize both c-semirings and positive structures. In fact, when in a bipolar structure $\Box = \top$, we have a c-semiring and, when $\Box = \bot$, we have a positive structure. In the following, we will write $+_n$ instead of $+_{|N}$ and $+_p$ instead of $+_{|P}$. Similarly for $\times_n$ and $\times_p$. When operator $\times$ will be applied to a pair in $(N \times P)$, we will sometimes write $\times_{np}$ and we will call it compensation operator. Given the way the ordering is induced by $+$ on $N \cup P$, easily, we have $\bot \leq \Box \leq \top$. Thus, there is a unique maximum element (that is, $\top$), a unique minimum element (that is, $\bot$); the element $\Box$ is smaller than any positive preference and greater than any negative preference, and it is used to model indifference. A bipolar preference structure allows us to have a richer structure for one kind of preference, that is common in real-life problems. In fact, we can have different lattices $(P, \leq_p)$ and $(N, \leq_n)$. For example, we could be satisfied with just two levels of negative preferences, while requiring several levels of positive preferences.

It is easy to show that the combination of a positive and a negative preference is a preference which is higher than, or equal to, the negative one and lower than, or equal to, the positive one. The following theorems hold when a bipolar preference structure $(N, P, +, \times, \bot, \Box, \top)$ is given.

Theorem 2. *For all $p \in P$ and $n \in N$, $n \leq p \times n \leq p$.*

This means that the compensation of positive and negative preferences must lie in one of the chains between the two combined preferences. Notice that all such chains pass through the indifference element $\Box$. Possible choices for combining strictly positive with strictly negative preferences are thus the average, the median, the min or the max operator. Moreover, by monotonicity, if $\top \times \bot = \bot$, then $\forall p \in P$, $p \times \bot = \bot$. Similarly, if $\top \times \bot = \top$, then $\forall n \in N$, $n \times \top = \top$.

In general, the compensation operator $\times$ may be not associative. Here we list some sufficient conditions for the non-associativity of the $\times$ operator.

Theorem 3. *The operator $\times$ is not associative if:*

- $\top \times \bot = c \in (N \cup P) - \{\top, \bot\}$, *or*
- $\exists p \in P - \{\top\}$ *and* $n \in N - \{\bot\}$ *s. t.* $p \times n = \Box$ *and at least one of the following conditions holds,*
  - $\times_p$ *or* $\times_n$ *is idempotent;*
  - $\exists p' \in P - \{p, \top\}$ *s. t.* $p' \times n = \Box$ *(or* $\exists n' \in N - \{n, \bot\}$ *s. t.* $p \times n' = \Box$*);*
  - $\top \times \bot = \bot$ *and* $\exists n' \in N - \{\bot\}$ *s. t.* $n \times n' = \bot$*;*
  - $\top \times \bot = \top$ *and* $\exists p' \in P - \{\top\}$ *s. t.* $p \times p' = \top$*;*
  - $\exists a, c \in N \cup P$ *s. t.* $a \times p = c$ *iff* $c \times n \neq a$*; (or s. t.* $a \times n = c$ *iff* $c \times p \neq a$*).*

These sufficient conditions refer to various aspects of a bipolar preference structure: properties of the operators, cardinality of the equivalence classes, shape of the orderings of P and N, the relation between $\times$ and the other operators. Since some of these conditions often occur naturally in practice, it is not reasonable to require associativity of $\times$.

In the following table each row corresponds to a bipolar preference structure.

N,P	$+_p$, $\times_p$	$+_n$, $\times_n$	$\times_{np}$	$\bot$, $\Box$, $\top$
R^-, R^+	max, sum	max, sum	sum	$-\infty$, 0, $+\infty$
$[-1, 0]$, $[0, 1]$	max, max	max, min	sum	-1, 0, 1
$[0, 1]$, $[1, +\infty]$	max, prod	max, prod	prod	0, 1, $+\infty$

In the first structure positive preferences are positive real numbers and negative preferences are negative real numbers, the compensation is given by sum, while the ordering is given by max. In the second structure positive preferences are between 0 and 1 and negative preferences between -1 and 0. Again, compensation is sum, and the order is given by max. In the third structure positive preferences are between 1 and $+\infty$ and negative preferences between 0 and 1. Compensation is obtained by multiplying the preferences and ordering is again via max. If $\top \times \bot \in \{\top, \bot\}$, then compensation in the first and in the third structure is associative.

6 Bipolar preference problems

Once we have defined bipolar preference structures, we can define a notion of bipolar constraint, which is just a constraint where each assignment of values to its variables is associated to one of the elements in a bipolar preference structure.

Definition 3. *Given a bipolar preference structure* $(N, P, +, \times, \bot, \Box, \top)$ *a finite set* D *(the domain of the variables), and an ordered set of variables* V*, a constraint is a pair* $\langle def, con\rangle$ *where* $con \subseteq V$ *and* $def : D^{|con|} \rightarrow (N \cup P)$*.*

A bipolar CSP (V, C) is then just a set of variables V and a set of bipolar constraints C over V. We propose a way of defining the optimal solutions of a bipolar CSP that avoids problems due to the possible non-associativity.

Definition 4. *A solution of a bipolar CSP* (V, C) *is a complete assignment to all variables in* V*, say* s*, with an associated preference* $pref(s) = (p_1 \times_p \ldots \times_p p_k) \times (n_1 \times_n \ldots \times_n n_l)$*, where, for* $i := 1, \ldots, k$ $p_i \in P$*, for* $j := 1, \ldots, l$ $n_j \in N$*, and* $\exists \langle def, con\rangle \in C$ *such that* $p_i = def(s \downarrow_{con})$ *or* $n_j = def(s \downarrow_{con})$*. A solution* s *is an optimal solution if there is no other solution* s' *with* $pref(s') > pref(s)$*.*

In this definition, the preference of a solution s is obtained by combining all the positive preferences associated to its projections over the constraints, combining all the negative preferences associated to its projections over the constraints, and then, combining the two preferences obtained so far. If $\times$ is associative, then other definitions of solution preference could be used while giving the same result.

6.1 Solving bipolar CSPs

Bipolar problems are NP-hard, since they generalise both classical and soft constraints, which are already difficult problems. However, we can devise algorithms and heuristics to solve them, hopefully efficiently in the average case. Preference problems based on c-semirings can be solved via a branch and bound technique, possibly augmented via soft constraint propagation, which may lower the preferences and thus allow for the computation of better bounds [2]. In bipolar CSPs, we have positive and negative preferences. Branch and bound techniques can be adapted to compute, at each search node k, an upper bound ub to the preferences of all the solutions in the k-rooted subtree.

If $\times$ is non-associative, then each node is associated to a positive and a negative preference, say p and n, which is obtained by aggregating all preferences of the same type in the instantiated part of the problem. An upper bound for the subtree can be computed, for example, by taking the aggregation of all the best positive and negative preferences in the non-instantiated part of the problem, say p' and n', and by aggregating them to the positive and negative preferences of the current node. This produces the upper bound $ub = (p \times_p p') \times (n \times_n n')$, where $p' = p_1 \times_p \ldots \times_p p_s$, $n' = n_1 \times_n \ldots \times_n n_w$, and $r = s + w$ is the number of non-instantiated variables/constraints. Thus ub can be computed via $r - 1$ aggregation steps and one compensation step.

If $\times$ is associative, however, we don't need to postpone compensation until all constraints have been considered, but we can interleave compensation and aggregation while searching for an optimal solution. This means that we can keep just one value $v = p \times n$ for each search node, that can be positive or negative, which is obtained by aggregating all preferences (both positive and negative) obtained in the instantiated part of the problem. The same can be done considering the best preferences in the uninstantiated part of the problem, obtaining a value v'. Thus, ub can now be written as $ub = v \times v'$, where $v' = a_1 \times \ldots \times a_r$, where $a_i \in N \cup P$ is the best preference found in a constraint of the uninstantiated part of the problem. Thus now ub can be computed via at most $r-1$ steps among which there can be many compensation steps. A compensation can generate the indifference $\square$, which is the unit element for the compensation operator. Thus, when $\square$ is generated, the successive computation step can be avoided.

If $ub \leq v$, where v is the preference of the best solution so far, we can prune the k-rooted subtree. To improve this upper bound, we can propagate negative preferences as it is done in soft constraints [2, 4]. In fact, such a propagation may lower the negative values while not changing the semantics of the problem. Due to the monotonicity of $\times$ and $\times_n$, the upper bound may thus become smaller and allow for more pruning. Positive preference can be propagated as well. However, since $\times_p$ returns higher positive preferences, their propagation produces higher values. This is not helpful in improving the upper bound, since monotonicity of $\times$ implies that a higher upper bound is obtained.

7 Related and future work

Bipolar reasoning and preferences have recently attracted some interest in the AI community. In [1], a bipolar preference model based on a fuzzy-possibilistic approach is described, but positive and negative preferences are kept separate and no compensation is allowed. In [7] totally ordered unipolar and bipolar preference scales are used, whereas we have presented a way to deal with partially ordered bipolar scales.

We plan to develop a solver for bipolar CSPs, which should be flexible enough to accommodate for both associative and non-associative compensation operators. We also intend to consider the presence of uncertainty in bipolar problems, possibly using possibility theory and to develop solving techniques for such scenarios. Another line of future research is the generalization of other preference formalisms, such as multicriteria methods and CP-nets, to deal with bipolar preferences and to study the relation between bipolarization and importance tredeoffs.

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